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Sufficiently Euclidean Banach Spaces and Fully Nuclear Operators.

Charles Patrick Stegall Jr

Louisiana State University and Agricultural & Mechanical College

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SUFFICIENTLY EUCLIDEAN BANACH SPACES AND FULLY NUCLEAR OPERATORS

A Dissertation

**Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy**

in

The Department of Mathematics

by

**Charles Patrick Stegall, Jr.
B.S., Louisiana State University, 1965
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ABSTRACT

The paper is devoted to a study of the conjecture of A. Grothendieck that if E and F are Banach spaces and all operators from E to F are nuclear, then E or F must be finite dimensional. A partial solution is given to this conjecture. Two new properties of Banach spaces are introduced. We call the properties sufficiently Euclidean and Property S. It is shown that the classical reflexive spaces are sufficiently Euclidean.

A class of operators between two Banach spaces is defined, called the fully nuclear operators and a determination of the structure of this class in certain cases is made. The principal result is that if all the continuous linear operators from a Banach space E to a Banach space F are fully nuclear then one of E, F must be finite dimensional.

Characterization of \mathfrak{L}_∞ and \mathfrak{L}_2 spaces are given in terms of fully nuclear operators. An immediate consequence of our results is that an infinite dimensional $\mathfrak{L}_1(\mathfrak{L}_\infty)$ space contains a subspace with a Schauder basis that is not an $\mathfrak{L}_1(\mathfrak{L}_\infty)$ space.

INTRODUCTION

The notation used will be that of [29] where one can find the basic theorems of topological vector spaces that we shall use. We shall be considering only normed spaces, and for the most part only Banach spaces.

This work is based on the Memoir of Grothendieck [7] (see also [8], [9], [30], [23]). No attempt will be made to give proofs, or even detailed statements of the theorems of [7] that we shall be using. We shall list the theorems basic to this work.

The word operator (and sometimes map) will mean a bounded linear transformation. We shall denote by $\mathcal{L}(E, F)$ the operators from E to F . By an isomorphism, we shall mean a one-to-one operator that is open. We shall usually say whether the isomorphism is onto or not; isometry has the usual meaning. If $T: E \rightarrow F$ is an operator, the range of T is F and the image of T is the subset $T(E)$ of F . A projection P is an operator in $\mathcal{L}(E, E)$ such that $P^2 = P$. If $T: E \rightarrow F$ is an operator and $E_0 \subseteq E$ we shall denote by $T|_{E_0}$ the restriction of T to E_0 ; if $T(E) \subseteq G \subseteq F$ then by the restriction of T to G we mean the operator $T_0: E \rightarrow G$.

If $\{x_\alpha\} \subseteq E$ where E is a Banach space, then by $[x_\alpha]$ we mean the closed, linear span of $\{x_\alpha\}$ in E ; that is, the smallest closed subspace of E containing $\{x_\alpha\}$.

We begin by giving the following definition:

Definition: If E and F are isomorphic Banach spaces, the distance coefficient of E and F , $d(E, F)$ is defined by

$$d(E, F) = \inf \{ \|T\| \cdot \|T^{-1}\| : T: E \rightarrow F \\ T \text{ is an onto isomorphism} \}$$

We shall denote by c_0 , ℓ_p , $1 \leq p \leq \infty$, the usual sequence spaces with their standard norm. By ℓ_p^n , $1 \leq p \leq \infty$, $n = 1, 2, 3, \dots$ we shall mean the n -fold product of the complex (or real) numbers with the following norm:

$$\text{For } 1 \leq p < \infty, \|(t_1, \dots, t_n)\| = \left(\sum_{i=1}^n |t_i|^p \right)^{1/p}$$

$$\text{For } p = \infty, \|(t_1, \dots, t_n)\| = \max_{1 \leq i \leq n} |t_i|.$$

As usual, we shall denote by $L_p(S, \mu)$, $1 \leq p \leq \infty$, the space of p -integrable functions given by a measure μ (in the sense of Bourbaki) on a locally compact Hausdorff space S (actually equivalence classes of such functions!).

Another class of spaces which we shall be using are the \mathfrak{f}_p spaces of [18]. A space is an $\mathfrak{f}_{p, \lambda}$ space if for each finite dimensional $F \subseteq \mathfrak{f}_{p, \lambda}$ there exists a finite dimensional subspace E with $F \subseteq E \subseteq \mathfrak{f}_{p, \lambda}$, such that $d(E, \ell_p^n) \leq \lambda$ where $n = \dim(E)$, the dimension of E . We shall often use the result that the dual of an \mathfrak{f}_p is an \mathfrak{f}_q space where $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ [18], [19].

The projection constant $P(E)$ of a space E is defined by:

$$P(E) = \sup_F \inf \{ \|P\| : P:F \rightarrow E \text{ is a projection} \}$$

where the supremum is taken over all spaces F that have a subspace linearly isometric to E . Note that if $\dim(E) = n$, then $P(E) \leq n$ [28].

A P space (injective space, P_λ space for some λ) is a space whose projection constant is finite [15].

A series $\sum_{i=1}^{\infty} x_i$ in a Banach space is said to converge absolutely if $\sum_{i=1}^{\infty} \|x_i\| < +\infty$ and to converge unconditionally if $\sum_{i=1}^{\infty} \epsilon_i x_i$ converges for any choice of complex numbers ϵ_i , $|\epsilon_i| = 1$ (or equivalently, if the unordered partial sums converge).

By a basis we will mean a Schauder basis in the sense of [1], and an unconditional basis is a basis in which each representation converges unconditionally. The basis constant is the supremum of the norms of the partial sum operators.

An operator $T:E \rightarrow F$ is said to be absolutely summing if for each unconditionally convergent series $\sum_{i=1}^{\infty} x_i$ in E , the series $\sum_{i=1}^{\infty} Tx_i$ is absolutely convergent in F [7, p. 155] (also [24] for generalizations and other characterizations). By $AS(E,F)$ we shall denote the absolutely summing operators from E to F .

In references to the work of Grothendieck, we sometimes give references to sources other than Grothendieck. This is done as a matter of convenience to the author and reader.

We now give the definition of a quasi-nuclear operator [23], [25]. An operator $T: E \rightarrow F$ is said to be quasi-nuclear if there exists an absolutely convergent series $\sum_{i=1}^{\infty} f_i$ in E' such that $\|Tx\| \leq \sum_{i=1}^{\infty} |f_i(x)|$ for each x in E . We denote by $QN(E, F)$ the quasi-nuclear operators from E to F .

On the algebraic tensor product $E \otimes F$ of two Banach spaces we shall be considering only two topologies. The first, the projective (π , greatest cross-norm) topology [7, § 1, no. 1] is defined as follows:

$$\text{For } u \in E \otimes F, \|u\| = \inf \left\{ \sum_{i=1}^n \|x_i\| \cdot \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}$$

An element u in $E \hat{\otimes} F$ may be represented in the following form [7, § 2, no. 1]

$$u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i,$$

where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, $x_i \rightarrow 0$, $y_i \rightarrow 0$.

Theorem: The dual of the Banach space $E \hat{\otimes} F$ is the space of continuous bilinear forms on $E \times F$, which is the same as $\mathcal{L}(E, F')$ or $\mathcal{L}(F, E')$, with the duality given by the following: if $u = \sum_{i=1}^n x_i \otimes y_i$ is in $E \otimes F$, and T is in $\mathcal{L}(E, F')$ then $\langle u, T \rangle = \langle \sum_{i=1}^n x_i \otimes y_i, T \rangle = \sum_{i=1}^n \langle Tx_i, y_i \rangle$.

The second, the inductive (ϵ , least cross-norm) topology [7, § 3, no. 1] on $E \otimes F$ is defined as follows:

$$\text{For } u = \sum_{i=1}^n x_i \otimes y_i$$

$$\|u\| = \sup_{\substack{g \in F', \|g\|=1 \\ f \in E', \|f\|=1}} \left| \sum_{i=1}^n f(x_i)g(y_i) \right|.$$

The inductive topology on $E \otimes F$ is the topology induced on $E \otimes F$ by considering it as a subspace of $\mathcal{L}(E', F)$. We shall denote by $E \hat{\otimes} F$ the completion of $E \otimes F$ in the inductive topology.

Theorem: [7, § 3, § 4] [30, expose 7] The dual of the Banach space $E \hat{\otimes} F$ is the space of integral bilinear forms on $E \times F$, which will be denoted by $J(E, F)$. An operator $T: E \rightarrow F$ is integral if it induces an integral bilinear form on $E \times F'$.

It follows that if $u \in E \otimes F$ then $\|u\|_e \leq \|u\|_\pi$, thus there is induced an operator $J: E \hat{\otimes} F \rightarrow E \hat{\otimes} F$ which shall be called the canonical operator. An operator $T: E \rightarrow F$ is said to be nuclear if there exists a u in $E' \hat{\otimes} F$ such that $J(u) = T$, where $J: E' \hat{\otimes} F \rightarrow E' \hat{\otimes} F$ is the canonical operator. In other words if there are sequences (λ_i) in ℓ_1 , $\{f_i\} \subseteq E'$ with $f_i \rightarrow 0$, $\{y_i\} \subseteq F$ with $y_i \rightarrow 0$ such that $Tx = \sum_{i=1}^{\infty} \lambda_i f_i(x) y_i$. We shall denote by $N(E, F)$ the nuclear operators from E to F .

If $E_0 \subseteq E$ and $F_0 \subseteq F$ then there is the obvious operator $J: E_0 \otimes F_0 \rightarrow E \otimes F$. In the projective topology, the adjoint J' is the restriction of a bilinear form on $E \times F$ to a bilinear form on $E_0 \times F_0$; and in the inductive topology J' is onto (because J is an isometry), and is the restriction of an integral bilinear form on $E \times F$ to an integral bilinear form on $E_0 \times F_0$.

We state now some consequences of the approximation property and of the metric approximation property [7, § 5]. A Banach space E has the approximation property if it satisfies either of the two following equivalent conditions:

(I) For every Banach space F , the canonical operator $J: E \hat{\otimes} F \rightarrow E \hat{\otimes} F$ is one-to-one.

(II) For each Banach space F the operator $I: E \hat{\otimes} F \rightarrow J(E', F')$ is one-to-one, where I is defined as follows: if $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$ is in $E \hat{\otimes} F$, and $v = \sum_{j=1}^n f_j \otimes g_j$ is in $E' \otimes F'$ and $\|v\|_e \leq 1$, then

$$\begin{aligned} |\langle v, Iu \rangle| &= \left| \left\langle \sum_{j=1}^n f_j \otimes g_j, \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i \right\rangle \right| \\ &= \left| \sum_{i=1}^{\infty} \left(\sum_{j=1}^n \lambda_i f_j(x_i) g_j(y_i) \right) \right| \\ &\leq \sum_{i=1}^{\infty} |\lambda_i| \cdot \|x_i\| \cdot \|y_i\| \end{aligned}$$

which shows that I is a well defined, bounded operator. There are many other equivalent statements of the approximation property.

Concerning the metric approximation property, we shall need the following theorem: if E has the metric approximation property, then the operator $I: E \hat{\otimes} F \rightarrow J(E', F')$ is an into isometry for each space F [7, p. 181, Corollaire 1, § 2].

We shall establish the following result which is used in the proof of Theorem II. 11.

Theorem: Suppose E has the metric approximation property and F is a complemented subspace of E . Then for each Banach space G , the canonical operator $I: F \hat{\otimes} G \rightarrow J(F', G')$ is an into isomorphism.

Proof: Suppose $P:E \rightarrow E$ is a projection such that $P(E) = F$. Consider the following diagrams; where $I:F \rightarrow E$ is inclusion:

$$F' \hat{\otimes} G' \xrightarrow{\Psi_1} E' \hat{\otimes} G' \xrightarrow{\Psi_2} F' \hat{\otimes} G'$$

$$\mathfrak{L}(F, G') \xrightarrow{\Phi_1} \mathfrak{L}(E, G') \xrightarrow{\Phi_2} \mathfrak{L}(F, G')$$

$$\Psi_1(y' \otimes z') = P'y' \otimes z', \quad \Psi_2(x' \otimes z') = I'y' \otimes z'.$$

If $T \in \mathfrak{L}(F, G')$ then $\Phi_1(T) = TP$. If $S \in \mathfrak{L}(E, G')$ then $\Phi_2(S)$ is the restriction of S to F . Clearly Φ_1 is an isomorphism into, Φ_2 is onto, and $\Phi_2\Phi_1$ is the identity. Similarly Ψ_1 is an isomorphism into, Ψ_2 is onto, and $\Psi_2\Psi_1$ is the identity. Consider the following diagram:

$$\begin{array}{ccccc} F \hat{\otimes} G & \xrightarrow{J_2} & E \hat{\otimes} G & \xrightarrow{J_1} & F \hat{\otimes} G \\ I_1 \downarrow & & I_2 \downarrow & & I_1 \downarrow \\ J(F', G') & \xleftarrow{\Psi'_1} & J(E', G') & \xleftarrow{\Psi'_2} & J(F', G') \end{array}$$

We obviously have $J'_2 = \Phi_2$, $J'_1 = \Phi_1$, and we have that J_1 and Ψ'_1 are onto and J_2 and Ψ'_2 are into isomorphisms. We know that I_2 is an into isometry. We have, for u in $F \hat{\otimes} G$, $c||u|| \leq ||J_2u|| \leq C||u||$, and for v in $J(F', G')$ $m||v|| \leq ||\Psi'_2v|| \leq M||v||$. Thus we have the following:

$$m||I_1u|| \leq ||\Psi'_2I_1u|| \leq M||I_1u||$$

$$c||u|| \leq ||I_2J_2u|| = ||J_2u|| \leq C||u||$$

$$I_2J_2u = \Psi'_2I_1u$$

Hence it follows that $c||u|| \leq M||I_1u||$ or $c/M||u|| \leq ||I_1u||$.

This says that I_1 is an isomorphism.

I. SUFFICIENTLY EUCLIDEAN BANACH SPACES

In this chapter we introduce a class of Banach spaces having finite dimensional subspaces of a certain type. We make this precise below.

One of the most profound results in Banach space theory is the following theorem of A. Dvoretzky [4] concerning spherical sections of convex bodies in Banach spaces.

Theorem 1.1 (Dvoretzky): For each $\epsilon > 0$ and each positive integer n , there exists a positive integer $n(\epsilon)$ such that if E is any Banach space and the dimension of E is greater than $n(\epsilon)$, then there exists a subspace F of E such that $d(F, \ell_2^n) \leq 1 + \epsilon$.

Thus, in each infinite dimensional Banach space, there are finite dimensional subspaces of arbitrarily large dimension, nearly isometric to Euclidean spaces. The property we need is somewhat stronger.

Definition 1.2: A Banach space E is sufficiently Euclidean if there is a positive constant C and sequences of operators $\{J_n\}$, $\{P_n\}$ such that

$$\ell_2^n \xrightarrow{J_n} E \xrightarrow{P_n} \ell_2^n$$

and $P_n J_n = I_n$, the identity operator on ℓ_2^n , and $\|P_n\| \cdot \|J_n\| \leq C$.

Remark 1.3 (1): Obviously we may assume $\|P_n\| = 1$, and $1 \leq \|J_n\| \leq C$ for each n .

(ii): It is immediate that any space isomorphic to a Hilbert space (i.e., any ℓ_2 space) is sufficiently Euclidean.

(iii): If F is sufficiently Euclidean and complemented in E , then E is sufficiently Euclidean.

Let us compare Definition I.2 with Theorem I.1. While Dvoretzky's theorem says that in an infinite dimensional Banach space there are many, nearly isometric copies of ℓ_2^n for arbitrarily large n , it says nothing about the norm of a projection onto one of these subspaces. For a space to be sufficiently Euclidean we are requiring that there be a sequence of finite dimensional subspaces of increasing dimension such that each is "uniformly isomorphic" to ℓ_2^n for the appropriate n , and the projections onto these spaces are uniformly bounded.

From the definition, it is not immediately clear which Banach spaces are sufficiently Euclidean. It might appear that the only such spaces are those that isomorphically have Hilbert space as a complemented subspace. In fact, the class of sufficiently Euclidean spaces is rather large, and contains the classical reflexive spaces, as well as some hereditarily non-reflexive spaces.

We have the following theorem which follows immediately from the definition and the result of Zippin [34].

Theorem I.4: A Banach space E is sufficiently Euclidean if and only if E' is also sufficiently Euclidean.

Proof: If E is sufficiently Euclidean, with C the constant and $\{P_n\}$ and $\{J_n\}$ the given operators, then clearly the adjoints $\{P'_n\}$, $\{J'_n\}$ and the

same constant C will suffice for E' . The converse follows immediately from Zippin's version of the Principle of Local Reflexivity [19], which we state: If G is a finite dimensional subspace of E'' , and $P:E'' \rightarrow G$ is a projection, then for each $\epsilon > 0$, there exists a subspace G_0 of E , and an isomorphism $S:G \rightarrow G_0$ with S restricted to $G \cap E$ equal to the identity, and $\|S\| \cdot \|S^{-1}\| \leq 1 + \epsilon$ and there is a projection $Q:E \rightarrow G_0$ such that $\|Q\| \leq (1 + \epsilon)\|P\|$ [34]. In fact, if E' is sufficiently Euclidean, then E'' is also, so we have the operators $\{J_n\}$, $\{P_n\}$ and the constant C given by the definition, Let $\epsilon = 1$, $J_n P_n$ be the projection, $J_n P_n(E'')$ the finite dimensional subspace of E'' , then we obtain an isomorphism $S_n:J_n P_n(E'') \rightarrow E$, and a projection $Q_n:E \rightarrow E$ such that $Q_n(E) = S_n:J_n P_n(E'')$ with $\|S_n\| \cdot \|S_n^{-1}\| \leq 2$ and $\|Q_n\| \leq 2C$, then $\{S_n P_n\}$ and $\{J_n S_n^{-1} Q_n\}$ are the desired operators and $4C^2$ the desired constant.

Theorem I.5: For $1 < p < \infty$, ℓ_p is sufficiently Euclidean.

Proof: The following proof is essentially contained in [21]. Let γ_i denote the i th Rademacher function, that is $\gamma_i(t) = \text{sgn} \sin(2^{i-1} \pi t)$, and consider $R_n = [\gamma_1, \dots, \gamma_n] \subseteq L_p[0,1]$. Let $G_{2^n} \subseteq L_p[0,1]$ be the space spanned by $\chi\left[\frac{K-1}{2^n}, \frac{K}{2^n}\right]$, $K = 1, \dots, 2^n$, $n = 1, 2, 3, \dots$. By extending the map $e_K \rightarrow \chi\left[\frac{K-1}{2^n}, \frac{K}{2^n}\right]$, where e_K is the K th unit vector of $\ell_{2^n}^p$, it is easy to see that the resulting operator $T_n:\ell_p^{2^n} \rightarrow G_{2^n}$ is an isometry. By the definition of γ_i we have $R_n \subseteq G_{2^n}$ for each n and, by the classical Khinchin inequality, R_n is isomorphic to ℓ_2^n and the norm of the isomorphism, C_p , depends only on p . Define

$Q_n(x) = \sum_{i=1}^n \left[\int_0^1 x(t) \gamma_i(t) dt \right] \gamma_i$. Clearly $Q_n: L_p[0,1] \rightarrow R_n$, and by the orthogonality properties of the Rademacher system it follows that $Q_n^2 = Q_n$ for each n , that is Q_n is a projection of $L_p[0,1]$ onto R_n for each n , and $\|Q_n\| \leq M_p$, a constant depending only on p [13, p. 245]. (This much of the proof shows that $L_p[0,1]$ is sufficiently Euclidean for $1 < p < \infty$). Let $S_n: G_2^n \rightarrow R_n$ be the restriction of Q_n to G_2^n and let $U_n: \ell_p \rightarrow \ell_p^{2^n}$ be the natural projection. Then $P_n = T_n^{-1} S_n^T U_n$ is a projection of ℓ_p onto a subspace whose distance from ℓ_2^n is no more than C_p . Hence, we have that ℓ_p is sufficiently Euclidean.

From [18] it follows that each ℓ_p space, $1 \leq p < \infty$, has a complemented subspace isomorphic to ℓ_p . Thus we have the following result.

Corollary 1.6: For $1 < p < \infty$, an ℓ_p space is sufficiently Euclidean.

The cases $p = 1$ or $p = \infty$ will be considered below.

It is well known [1] and [2], that for $p \neq 2$, ℓ_p contains no isomorphic copy of ℓ_2 .

We introduce another class of Banach spaces, which as we shall see, properly contains the sufficiently Euclidean spaces.

Definition 1.7: A Banach space E is said to have Property (S) if there exists unconditionally convergent series $\sum_{i=1}^{\infty} f_i$ in E' , $\sum_{j=1}^{\infty} x_j$ in E , such that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|^2} \right)^{\frac{1}{2}} = \infty.$$

We see the importance of spaces with Property (S) in the following theorem.

Theorem 1.8: Let E be a Banach space with Property (S), and let F be any Banach space. Then there is an operator $T:E \rightarrow F$ that is not absolutely summing.

Proof: It follows from [4] and [18] that if $\mathcal{L}(E,F) = AS(E,F)$ then $\mathcal{L}(E, \ell_2) = AS(E, \ell_2)$. We shall prove there is an operator $T:E \rightarrow \ell_2$ that is not absolutely summing. Let $\sum_{i=1}^{\infty} f_i, \sum_{i=1}^{\infty} x_i$ be the series which exists by Definition 1.7. Let $\{e_j\}$ be the usual orthonormal basis of ℓ_2 . Define $T:E \rightarrow \ell_2$ by $T(x) = \sum_{j=1}^{\infty} ||f_j||^{-\frac{1}{2}} f_j(x) e_j$. If $C = \sup_{||x||=1} \sum_{j=1}^{\infty} |f_j(x)|$, which is finite because $\sum_{j=1}^{\infty} f_j$ is unconditionally convergent, then for $||x|| \leq 1$,

$$\begin{aligned} ||Tx|| &= \left\| \sum_{j=1}^{\infty} \frac{f_j(x)}{||f_j||^{\frac{1}{2}}} e_j \right\| \\ &= \left(\sum_{j=1}^{\infty} \frac{|f_j(x)|^2}{||f_j||} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=1}^{\infty} |f_j(x)| \right)^{\frac{1}{2}} \\ &\leq \sqrt{C}. \end{aligned}$$

Thus T is a bounded operator. Also, we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n ||Tx_i|| = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{||f_j||} \right)^{\frac{1}{2}} = \infty.$$

That is, T is not absolutely summing because the image $\{Tx_i\}$ of the unconditionally convergent series $\{x_i\}$ is not absolutely convergent.

We now determine the relationship between sufficiently Euclidean Banach spaces and those with Property (S).

Theorem 1.9: A sufficiently Euclidean Banach space E has Property (S).

Proof: Choose $\{t_K\}$ in ℓ_4 but not in ℓ_2 and $t_K > 0$. Block $\{t_K\}$ such that

$$\sum_{j=1}^{\infty} \left(\sum_{K=n_j+1}^{n_{j+1}} t_K^4 \right)^{\frac{1}{2}} = K < \infty, \text{ and}$$

$0 = n_1 < n_2 < \dots$, $n_j \rightarrow \infty$ as $j \rightarrow \infty$. Let $m_j = n_{j+1} - n_j$; if E is sufficiently Euclidean we must have operators $\{J_j\}$, $\{P_j\}$, and a constant $C > 0$ such that

$$\ell_2^{m_j} \xrightarrow{J_j} E \xrightarrow{P_j} \ell_2^{m_j}$$

where $P_j J_j$ is the identity on $\ell_2^{m_j}$ and $||P_j|| \cdot ||J_j|| \leq C$. If $\{e_{ji}\}$ denotes the usual basis of $\ell_2^{m_j}$ then it follows as in [4, main lemma] that if $J_j(e_{ji}) = x_{n_j+i}$ then $\sum_{i=1}^{\infty} t_i^2 x_i$ is unconditionally convergent, as is $\sum_{i=1}^{\infty} t_i^2 f_i$ where $f_{n_j+i} = P_j'(e_{ji})$, and we may assume

$||P_j|| = 1$, $||J_j|| \leq C$. Thus,

$$\begin{aligned} & \sum_{i=1}^n \left(\sum_{j=1}^{\infty} \frac{t_i^4 t_j^4 |f_j(x_i)|^2}{|t_j|^2 ||f_j||} \right)^{\frac{1}{2}} \\ & \geq \sum_{i=1}^n \frac{t_i^3 |f_i(x_i)|}{||f_i||} > \sum_{i=1}^n t_i^3 \end{aligned}$$

The $\lim_n \sum_{i=1}^n t_i^3 = \infty$, so we also have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|^2} \right)^{\frac{1}{2}} = \infty.$$

Thus E has property (S).

There are several corollaries to Theorems I.8 and I.9, and other applications will be given in Chapter II.

Corollary I.10: No \mathfrak{L}_1 or \mathfrak{L}_∞ space is sufficiently Euclidean.

Proof: By the profound result of Grothendieck [8] (also [18]) an operator T from an \mathfrak{L}_1 space to an \mathfrak{L}_2 space is absolutely summing. By Theorem I.8, an \mathfrak{L}_1 space does not have Property (S) hence is not sufficiently Euclidean. Since \mathfrak{L}_1 and \mathfrak{L}_∞ spaces are in duality [19], Theorem I.4 says that \mathfrak{L}_∞ spaces are not sufficiently Euclidean.

Corollary I.11: If E contains an isomorphic copy of c_0 , then E has Property (S). (Hence, most, and perhaps all, \mathfrak{L}_∞ spaces have Property (S) [19].)

Proof: Suppose $J: c_0 \rightarrow E$ is such that $m\|x\| \leq \|Jx\| \leq m\|x\|$. If we choose f_i in E' , $\|f_i\| \leq C$, $f_i(Je_j) = \delta_{ij}$ where $\{e_j\}$ is the usual basis of c_0 , then certainly $\sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^{1/3} Je_j$ and $\sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^{4/3} f_j$ are unconditionally convergent (the latter is even absolutely convergent). And we have the following:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\sum_{j=1}^{\infty} \frac{|f_j(Je_i)|^{\frac{1}{i}}}{\frac{1}{j} \|f_j\|} \right)^{2/3} \frac{1}{j}^{8/3} \Bigg)^{1/2} \\
& \geq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{\frac{1}{i}}{\|f_i\|} \\
& \geq \lim_{n \rightarrow \infty} \frac{1}{C} \sum_{i=1}^n \frac{1}{i} = \infty
\end{aligned}$$

Thus E has property S .

Remark 1.12: In Corollary 1.10 we prove that an \mathfrak{L}_∞ space is not sufficiently Euclidean by using the result of Grothendieck, Pelczynski-Lindenstrauss, and Lindenstrauss-Rosenthal ([8], [18] and [19]) showing that a \mathfrak{L}_1 space is not sufficiently Euclidean. It is interesting that another proof can be given avoiding use of the inequality of Grothendieck ([8], [18]) which is the main tool in the proof of the theorem that all operators from an \mathfrak{L}_1 space to an \mathfrak{L}_2 are absolutely summing. It is known that a Banach space E is an \mathfrak{L}_∞ space if and only if E'' is a P_λ space for some $\lambda \geq 1$ [19], [15]. Then by [15, Cor. 3, p. 16] if B is a finite dimensional subspace of \mathfrak{L}_∞ on which there is a projection of norm no more than M , then B is a $P_{\lambda M}$ space. If E were sufficiently Euclidean, C the constant and $\{E_n\}$ the sequence of subspaces given by Definition 1.2, then by [28, Lemma 1, p. 246]:

$$\frac{1}{C} P(\ell_2^n) \leq P(E_n) \leq CP(\ell_2^n)$$

where $P(E_n)$ denotes the projection constant of E_n [10]. By the above, E_n is a $P_{\lambda C^2}$ space for all n . Thus $P(\ell_2^n) \leq \lambda C^3$. By [28, p. 246] we have

$$P(\ell_2^n) = \frac{n \Gamma(\frac{1}{2}n)}{\sqrt{\pi} \Gamma(\frac{1}{2}n + \frac{1}{2})}$$

By Stirling's formula it easily follows that $P(\ell_2^n) \rightarrow \infty$. Thus E is not sufficiently Euclidean. Again applying [19] and theorem I.4 we obtain that an \mathcal{L}_1 space is not sufficiently Euclidean.

Although there are non sufficiently Euclidean spaces, we do have the following somewhat surprising result.

Theorem I.13: Every infinite dimensional Banach space E contains a sufficiently Euclidean subspace F . Moreover, F can be chosen to have a Schauder basis.

Proof: This theorem is essentially (of course, using different language) the result of Gurarii [11] in constructing basic sequences in Banach spaces. We outline the proof. If F and G are subspaces of a Banach space E then the inclination of F and G , $I(F, G)$, is defined by

$$I(F, G) = \inf \{ \|x + y\| : x \in F, \|x\| = 1, y \in G \}.$$

If $\{x_i\}$ is a sequence in E the index $\theta\{x_i\}$ of $\{x_i\}$ is defined by

$$\theta\{x_i\} = \inf_n \{d(S_n, [x_i : i > n])\}$$

where S_n denotes the unit sphere of $[x_i: i \leq n]$. It is well known [6] that a sequence is basic $\{x_i\}$ if and only if $\theta\{x_i\} > 0$. To prove Theorem I.13 we state the following three lemmas.

Lemma I. 14: Given $\epsilon > 0$, and a finite dimensional subspace F there is an infinite dimensional subspace G of E such that $I(F, G) > 1 - \epsilon$.

Proof: [33, pp. 29-30]

Lemma I. 15: Suppose the subspace $F_1, F_2 = G_1 \oplus G_2$ and F_3 satisfy

$$(i) \quad I(G_1 G_2) \geq \alpha > 0, \quad (ii) \quad I(F_1 \oplus F_2, F_3) \geq \beta > 0,$$

$$(iii) \quad I(F_1, F_2 \oplus F_3) \geq \beta > 0, \text{ then } I(F_1 \oplus G_1, G_2 \oplus F_3) \geq \frac{\alpha\beta^2}{2 + \alpha}$$

Proof: [33, p. 30]

Lemma I. 16: Let $\{x_i^1, x_i^2, \dots, x_i^{K_i}\} \ i = 1, 2, \dots$ be a basis for a K_i dimensional space $F_i \subseteq E$ and each i , for $\theta\{x_i^j\} \geq \alpha > 0, \ j = 1, \dots, K_i$, and for integers m, n with $m < n, I(P_1 \oplus \dots \oplus P_m, P_{m+1} \oplus \dots \oplus P_n) > \beta > 0$. Then $\{x_i^j\}$ is a basis for $[x_i^j]$ and $\theta\{x_i^j\} \geq \alpha\beta^2/2 + \alpha$.

Proof: [33, pp. 30-32].

Now by using Theorem I.1 we apply the above with $\{x_i^1, \dots, x_i^{K_i}\}$ corresponding under an ϵ -isometry to the unit vector basis of $\ell_2^{K_i}$. If $F = [x_i^j]$ then F is clearly sufficiently Euclidean. The projections from F onto $[x_i^1, \dots, x_i^{K_i}]$ have norm no greater than $2K$, where K is the basis constant. This proves Theorem I.13.

Corollary I.17: Every $\ell^1(\ell_\infty)$ space contains a subspace with a basis that is not an $\ell_1(\ell_\infty)$ space.

Proof: This follows immediately from Corollary I. 10 and Theorem I.13. This generalizes the examples given in [21], [16], for subspaces of c_0 and ℓ_1 respectively.

Corollary I. 18: There exists hereditarily non-reflexive sufficiently Euclidean spaces.

Proof: This is immediate from the well known result that c_0 and ℓ_1 have no reflexive subspaces.

We restate in somewhat different, and more useful, forms, some of the results we have obtained in this chapter.

Theorem I. 19: Let E and F be infinite dimensional Banach spaces. Then there is an infinite dimensional subspace E_0 of E and an operator $T:E_0 \rightarrow F$ that is not absolutely summing.

Proof: We have only to choose E_0 to be sufficiently Euclidean and apply Theorem I. 8.

Theorem I. 20: Let E and F be infinite dimensional Banach spaces. Then there is an infinite dimensional subspace F_0 of F and an operator $T:E \rightarrow F_0$ such that T' is not absolutely summing (hence T cannot be integral).

Proof: As in Theorem I. 9, we choose a sequence $\{t_k\}$ in ℓ_4 but not ℓ_3 and block (and reindex) such that

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{K_i} t_{ij}^4 \right)^{\frac{1}{4}} < \infty.$$

As in Theorem I. 13, choose a sufficiently Euclidean space $F_0 = [y_i^j]$, with corresponding basis constant $C/2$ and the "blocks" $\{y_i^1, \dots, y_i^{K_i}\}$ corresponding to the above blocking. By Theorem I. 1 find ϵ -isometric copies of $\ell_2^{K_i}$, say $[f_i^j]$ $j = 1, \dots, K_i$, $j = 1, 2, \dots$ in E' . Consider the operator

$$Tx = \sum_{i=1}^{\infty} \sum_{j=1}^{K_i} t_{ij} f_i^j(x) y_i^j.$$

For $\|x\| \leq 1$, we have

$$\begin{aligned} \|Tx\| &= \left\| \sum_{i=1}^{\infty} \sum_{j=1}^{K_i} t_{ij} f_i^j(x) y_i^j \right\| \\ &\leq \sum_{i=1}^{\infty} \left\| \sum_{j=1}^{K_i} t_{ij} f_i^j(x) y_i^j \right\| \\ &\leq C \cdot \sum_{i=1}^{\infty} \left(\sum_{j=1}^{K_i} t_{ij}^2 |f_i^j(x)|^2 \right)^{\frac{1}{2}} \\ &\leq C(1 + \epsilon)^{\frac{1}{2}} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{K_i} t_{ij}^2 |f_i^j(x)| \right)^{\frac{1}{2}} \\ &\leq C(1 + \epsilon)^{\frac{1}{2}} \sum_{i=1}^{\infty} \left(\sup_{\|x\|=1} \sum_{j=1}^{K_i} t_{ij}^2 |f_i^j(x)| \right)^{\frac{1}{2}} \\ &\leq C(1 + \epsilon)^{\frac{1}{2}} \sum_{i=1}^{\infty} \left(\sup_{|\epsilon_j|=1} \left\| \sum_{j=1}^{K_i} t_{ij}^2 \epsilon_j f_i^j \right\| \right)^{\frac{1}{2}} \\ &\leq C(1 + \epsilon) \sum_{i=1}^{\infty} \left(\sum_{j=1}^{K_i} t_{ij}^4 \right)^{\frac{1}{4}} \end{aligned}$$

Thus T is a bounded operator. Proceeding exactly as in Theorem I.8 it is seen that $T':F'_0 \rightarrow E'$ is not absolutely summing.

We end this chapter by proving the following

Theorem I. 21: A Banach space E is sufficiently Euclidean if and only if each subspace of finite co-dimension in E is sufficiently Euclidean.

Proof: It is immediate from Remark I. 3 (iii) that if E has a sufficiently Euclidean subspace of finite co-dimension, then E is sufficiently Euclidean. For the converse, let F be a subspace of co-dimension m , $1 \leq m < \infty$. If we have the constant C and the operators $J_n: \ell_2^n \rightarrow E$, $P_n: E \rightarrow \ell_2^n$ given by Definition I. 2, we must have for $n > m$ that $\dim(F \cap J_n(\ell_2^n)) > n - m$. There is a projection $Q_n: \ell_2^n \rightarrow P_n J_n(\ell_2^n)$ such that $\|Q_n\| = 1$. If we let S_n be the restriction of P_n to F , then $\{J_n Q_n\}$ and $\{S_n Q_n\}$ $n \geq m+1$ are the desired operators, and the same constant C will suffice.

Remark I. 22 (1): It follows from Theorem I. 21 that if $E = E^0$ is a quasi-reflexive space [3], $E^n = (E^{n-1})'$ and if there is a K such that E^K is sufficiently Euclidean, then it follows (without recourse to [34]) that E^n is sufficiently Euclidean for each n .

(ii): Let us mention that for every positive integer n there is a sufficiently Euclidean space which is quasi-reflexive of order n , namely the n -fold product of the space of James [12].

(iii): It is known that a Banach space E such that each separable subspace of E is isomorphic to Hilbert space is itself isomorphic to Hilbert space [17]. It is interesting to note that for $1 < p < \infty$ ℓ_p has the property that every subspace is sufficiently Euclidean. If E is an infinite dimensional subspace of ℓ_p , then it contains an isomorphic copy of ℓ_p , that is complemented in ℓ_p , hence in E [21]. Thus E is sufficiently Euclidean.

II. FULLY NUCLEAR OPERATORS

Of the many conjectures made by A. Grothendieck in his classical memoir [7] we find the following to be one of the most interesting:

(C) If E and F are Banach spaces and $\mathfrak{L}(E,F) = N(E,F)$ then E or F must be finite dimensional.

In a sense, the nuclear operators between two Banach spaces are the only "constructible" operators, that is, the operators of the most elementary form that include the finite dimensional operators.

To our knowledge this conjecture has only been considered in [7] and in the very important paper of Lindenstrauss and Pełczyński [18].

While we are not able to solve (C) in the generality asserted by Grothendieck we are able to give an affirmative answer to (C) under a somewhat more restricted hypothesis, which, however, does seem to include all the known cases.

We now define the concept of a fully nuclear operator.

Definition II.1: A bounded operator $T:E \rightarrow F$, E and F Banach spaces, is said to be fully nuclear if the restriction $T_0:E \rightarrow T(E)$ is nuclear.

Notation: $FN(E,F)$ will denote the fully nuclear operators from E to F .

Remark II.2 (1): Grothendieck [7, p. 40] gave a general method for obtaining nuclear operators that are not fully nuclear (using, of

course, different terminology); this construction is possible because of the following theorem due to Grothendieck [7, p. 40]:

Let F be a Banach space and E a closed subspace of F such that E is complemented in E'' . Then the canonical operator $J: E \hat{\otimes} E' \rightarrow F \hat{\otimes} E'$ is an into isomorphism if and only if E is complemented in F . The proof is immediate from the fact that J is an isomorphism if and only if $J': \mathcal{L}(F, E'') \rightarrow \mathcal{L}(E, E'')$ (J' is just restriction) is onto which is true if and only if E is complemented in F . Thus examples of nuclear, non-fully nuclear operators may be given by choosing a Banach space F with the approximation property, a closed, non complemented, reflexive subspace E with the approximation property. Then there exists an element T of $F \hat{\otimes} E'$ that is in the closure of $E \hat{\otimes} E'$ but not in $E \hat{\otimes} E'$. Regarding T as an operator, $T: E \rightarrow F$, then $T(E) \subseteq E$, T is nuclear, but the astriction $T_0: E \rightarrow E$ is not nuclear. Numerous examples are given later.

(ii): As is easily seen [23, p. 50] if F_0 is a dense subspace of F , $T: E \rightarrow F$. $T(E) \subseteq F_0$, $T_0: E \rightarrow F_0$ the astriction of T , then T is nuclear if and only if T_0 is nuclear. Thus in the definition of a fully nuclear operator, it does not matter whether we consider the astriction to the image or to the closure of the image.

(iii): Many of the difficulties in the theory of nuclear operators arise because the representation of a nuclear operator $T: E \rightarrow F$ depends on the range F and not the image $T(E)$. Obviously, these difficulties are not present when we consider fully nuclear operators.

(iv): Remark (i) shows that there are nuclear, non fully nuclear operators; we shall see that in a certain sense each nuclear operator can be obtained from a fully nuclear operator.

(v): If $T: E \rightarrow F$ is nuclear and the dimension of the kernel of T is infinite then $T = S - R$ where S and R are fully nuclear. We may assume T is in the form $\sum_{i=1}^{\infty} f_i \otimes y_i$, where $\sum_{i=1}^{\infty} \|y_i\| < \infty$ and $\|f_i\| \leq C$. Choose $\{x_i\}$ contained in the kernel of T and $\{g_i\} \subseteq E'$ such that $\|g_i\| = \|x_i\| = 1$, and $g_i(x_j) = \delta_{ij}$. Then $R = \sum_{i=1}^{\infty} g_i \otimes y_i$ and $S = \sum_{i=1}^{\infty} (f_i + g_i) \otimes y_i$ are fully nuclear and $T = S - R$.

The following obvious theorem will be used in what is to follow.

Theorem II. 3: Let $T: E \rightarrow F$ be nuclear and $\overline{T(E)}$ complemented in F . Then T is fully nuclear.

Proof: Since T is nuclear, there exists $\{x'_i\} \subseteq E'$, $\{y_i\} \subseteq F$, $\sum_{i=1}^{\infty} \|x'_i\| \cdot \|y_i\| < +\infty$ and $Tx = \sum_{i=1}^{\infty} \langle x, x'_i \rangle y_i$. If $P: E \rightarrow \overline{T(E)}$ is a bounded projection, then $Tx = PTx = \sum_{i=1}^{\infty} \langle x, x'_i \rangle Py_i$, and $\sum_{i=1}^{\infty} \|x'_i\| \cdot \|Py_i\| \leq \|P\| \sum_{i=1}^{\infty} \|x'_i\| \cdot \|y_i\| < +\infty$ and it follows immediately that T is fully nuclear.

Corollary II. 4: If F is isomorphic to a Hilbert space (in particular if F is finite dimensional) then for any Banach space E , $N(E, F) = FN(E, F)$.

Proof: The proof is immediate from the theorem since every closed subspace of F is complemented.

Our next result is a partial converse to II. 3.

Theorem II.5: Let F be a reflexive Banach space such that (i) each subspace of F has the approximation property and (ii) if $T:E \rightarrow F$ is nuclear then T is fully nuclear. Then every subspace of F is complemented.

Proof: Let F_0 be a closed subspace of F and J the canonical map of $F'_0 \hat{\otimes} F$ into $F'_0 \hat{\otimes} F$. J is one-to-one since F_0 has the approximation property [7, p. 165]. Moreover, J is closed. If not, by the construction given in II.2. (i), there would be a nuclear map $T:F_0 \rightarrow F$ with $T(F_0) \subseteq F_0$ such that the restriction $T_0:F_0 \rightarrow F_0$ is not nuclear, contradicting our hypothesis. J is an isomorphism into, thus J' is onto. $J':\mathfrak{L}(F, F''_0) \rightarrow \mathfrak{L}(F_0, F''_0)$ is restriction and since $F''_0 = F_0$, the identity map, $I:F_0 \rightarrow F_0$ has an extension $P:F \rightarrow F_0$ and $P^2 = P$, thus F_0 is complemented.

If H is a Hilbert space, and F is a closed subspace of H , and E is any Banach space then the canonical map $J:E' \hat{\otimes} F \rightarrow E' \hat{\otimes} H$ is an isometry.

The converse is also true.

Theorem II.6: Suppose F is a Banach space satisfying the conditions of II.6 and with the following property: the nuclear norm in $E' \hat{\otimes} \overline{T(E)}$ is the same as in $E' \hat{\otimes} F$. Then F is isometric to a Hilbert space.

Proof: As above, the canonical map $J: E'_0 \hat{\otimes} F_0 \rightarrow F'_0 \hat{\otimes} F$ is an isometry whenever F_0 is a closed subspace of F . Thus the adjoint map $J': \mathcal{L}(F, F_0) \rightarrow \mathcal{L}(F_0, F_0)$ is onto, and by the Hahn-Banach theorem, since J is an isometry there is an operator $P: F \rightarrow F_0$ such that P restricted to F_0 is the identity and $\|P\| = 1$. Thus by Kakutani's theorem [14], F is isometric to a Hilbert space.

The following remarks concerning II.5 and II.6 are interesting.

Remark II.7. (i): Grothendieck [7, § 5, sec. 2] has conjectured that all reflexive Banach space have the approximation property, so part (i) of the hypothesis of theorem II.5 is probably superfluous.

(ii): We have not used the full strength of hypothesis (ii) in II.5 in proving II.5 or II.6 since we used the properties only for subspaces of F . This does, however, give an easy, although apparently new, characterization of Hilbert space.

(iii): Using the construction of II.2. (i) and a recent result of Rosenthal we show how exacting II.5 and II.6 actually are. By Rosenthal [26, p. 52] there is an uncomplemented subspace of $L_{4/3}$ which is isomorphic to ℓ_2 . Let $S: \ell_2 \rightarrow L_{4/3}$ be such an into isomorphism. It follows as in II.2 (i) that $I \otimes S: \ell_2 \hat{\otimes} \ell_2 \rightarrow \ell_2 \hat{\otimes} L_{4/3}$ is not closed, that is, there is an operator $T: \ell_2 \rightarrow \ell_2$ that is not nuclear, but $ST: \ell_2 \rightarrow L_{4/3}$ is nuclear. This shows how important (ii) in the hypothesis of II.5 is.

(iv): If S and T have the meaning as in II.7. (iii) then as just observed, ST is nuclear; that is, we may compose a non-nuclear operator T with an into isomorphism S to obtain a nuclear operator.

In [23, p. 49], [25] Pietsch introduced the idea of a quasi-nuclear operator and proved that if $T:E \rightarrow F$ is quasi-nuclear, then if $J:F \rightarrow \ell_\infty(S)$, is the canonical operator where S is the unit ball of F' then JT is nuclear [23, p. 52]. Thus the situation in (iii) is actually quite common. In fact, it is interesting to note that the spaces in (iii) are reflexive. A larger class of examples is given in Corollary II. 13.

(v): It should be mentioned that II.5 may actually characterize Hilbert space. It is still an open question as to whether the conclusion of II.5 characterizes Hilbert space.

It is easily seen that if $T:E \rightarrow F$ is nuclear, then $T':F' \rightarrow E'$ is nuclear [23, p. 48]. The converse question is still open. ([23], some partial results are given in [7]). The analogous problem for fully nuclear operators is solved in the following theorem.

Theorem II.8: There is a non fully nuclear operator T whose adjoint T' is fully nuclear.

Proof: Let E be a sufficiently Euclidean subspace of ℓ_1 , with a basis, which exists by Theorem I. 13. Again we use the technique of II.2 (i). The canonical map $J:E \hat{\otimes} \ell_2 \rightarrow \ell_1 \hat{\otimes} \ell_2$ is one-to-one, since ℓ_2 has the approximation property [7, § 5, sec. 3]. J is not closed because if it were, the adjoint $J':\mathcal{L}(\ell_1, \ell_2) \rightarrow \mathcal{L}(E, \ell_2)$ which is restriction, would be onto. Every operator from ℓ_1 to ℓ_2 is absolutely summing [8], [18] and so the restriction of such an operator is absolutely summing. By Theorem I. 8 there exists a non absolutely

summing operator from E to l_2 . Thus $E \hat{\otimes} l_2$ is not closed in $l_1 \hat{\otimes} l_2$. Choose T in the closure of $E \hat{\otimes} l_2$ in $l_1 \hat{\otimes} l_2$ but not in $E \hat{\otimes} l_2$. Regarding T as an operator from l_2 to l_1 with $T(l_2) \subseteq E$, T is nuclear, but not fully nuclear. Also T' is nuclear and by Theorem II.3, fully nuclear, since $T': l_\infty \rightarrow l_2$.

Thus far in this chapter we have given some criteria for full nuclearity in terms of the range. We now develop criteria in terms of the domain. First we make an easy, though interesting, observation which seems to have received little attention.

Theorem II. 9: Let F be any Banach space, then any absolutely summing map $T: c_0 \rightarrow F$ is nuclear. In particular, any nuclear map from c_0 to F is fully nuclear.

Proof: Let (e_i) denote the usual unit vector basis of c_0 . The formal series $\sum_{i=1}^{\infty} e_i$ is weakly unconditionally Cauchy, and thus

$$\sum_{i=1}^{\infty} \|Te_i\| < +\infty \quad [23, p. 36]. \quad \text{If } x = \sum_{i=1}^{\infty} a_i e_i \text{ in } c_0, \text{ then } Tx =$$

$$\sum_{i=1}^{\infty} f_i(x) Te_i, \text{ where } (f_i) \text{ is the usual unit vector basis of } l_1. \text{ Also,}$$

$$\sum_{i=1}^{\infty} \|f_i\| \cdot \|Te_i\| = \sum_{i=1}^{\infty} \|Te_i\| < +\infty, \text{ which implies } T \text{ is nuclear.}$$

Also, if $T: c_0 \rightarrow F$ is nuclear, then it is absolutely summing, so the restriction $T_0: c_0 \rightarrow \overline{T(c_0)}$ is certainly absolutely summing, and by the above, nuclear. Thus T is fully nuclear.

Remark II. 10 (1): It is easily observed that the above theorem generalizes to $c_0(S)$ spaces, S any set. It is an interesting question as to whether Theorem II. 9 isomorphically characterizes $c_0(S)$ spaces.

A partial converse is given in Corollary II.14. Reasons for this conjecture are given in Remarks (ii) and (iii) and in Theorem II.11.

(ii): Grothendieck [8] has observed that the canonical operator from $C[0,1]$ to $L_1[0,1]$ is integral (hence absolutely summing) but not nuclear. By the profound result of Milutin [20] it follows that if K is compact metric and uncountable then there is a non-nuclear, integral operator $T:C(K) \rightarrow L_1[0,1]$.

(iii) More generally, the canonical operator $L_\infty(S, \mu) \rightarrow L_1(S, \mu)$, S a locally compact Hausdorff space, μ a measure on S [30, expose 3, 4, 16] is integral, but not, in general, nuclear (not in general, even compact) [30, expose 16, p. 5]. In particular, the operator $L_\infty[0,1] \rightarrow L_1[0,1]$ is integral but not nuclear. Moreover, \mathcal{L}_∞ is isomorphic to $L_\infty[0,1]$ [18]. Since \mathcal{L}_∞ is isometrically isomorphic to $C(\beta\mathbb{N})$, the Stone-Čech compactification of the positive integers \mathbb{N} , there is an integral, non-nuclear operator from $C(\beta\mathbb{N})$ to $L_1[0,1]$. It follows from a recent result of Rosenthal [27] that if K is any Stonian space (i.e., compact, Hausdorff and the closure of every open set is again open) then there is a non-nuclear, integral operator from $C(K)$ to $L_1[0,1]$.

The second part of Theorem II.9 is true for any \mathcal{L}_∞ space. We were led to this result by the fact that the technique of II.2 (i) fails because the dual of an \mathcal{L}_∞ space is an \mathcal{L}_1 space, and the dual of an \mathcal{L}_1 space is a P space [15], [18], [19].

Theorem II.11: A Banach space E is an \mathcal{L}_∞ space if and only if E' has the approximation property and $N(E, F) = FN(E, F)$ for each Banach space F .

Proof: Suppose $N(E, F) = FN(E, F)$ for each Banach space F . Consider the following diagram

$$\begin{array}{ccc} E' \hat{\otimes} F & \xrightarrow{J_1} & E' \hat{\otimes} F \\ I_1 \uparrow & & \uparrow I_2 \\ E' \hat{\otimes} F_0 & \xrightarrow{J_2} & E' \hat{\otimes} F_0 \end{array}$$

where F_0 is a closed subspace of F , I_1, I_2, J_1, J_2 are the canonical maps. If E' has the approximation property, J_1, J_2 are one-to-one, I_2 is always one-to-one, so I_1 is one-to-one. I_1 must be closed by II.2 (i). Since I_1 is closed, the adjoint map $I_1': \mathcal{L}(F, E'') \rightarrow \mathcal{L}(F_0, E'')$, which is restriction, must be onto. This says that E'' is an injective space, hence by [19] E is a \mathcal{L}_∞ space.

Suppose E is an $\mathcal{L}_{\infty, \lambda}$ space. Then E'' is an injective space [18], [19] and since E'' is complemented in some $\mathcal{L}_\infty(S)$ space, for some set S , E'' must have the approximation property, so E' must [30]. Let F_0 be a closed subspace of a Banach space F and consider the canonical map J

$$F' \hat{\otimes} E \xrightarrow{J} F'_0 \hat{\otimes} E.$$

This mapping is onto since E is an \mathcal{L}_∞ space [19]. Consider the following diagram, where all maps are the canonical ones:

$$\begin{array}{ccc} J(E, F') & \xleftarrow{J'} & J(E, F'_0) \\ L_1 \uparrow & & \uparrow L_2 \\ J(E'', F') & \xleftarrow{K_1} & J(E'', F'_0) \\ I_1 \uparrow & & \uparrow I_2 \\ E' \hat{\otimes} F & \xleftarrow{K_2} & E' \hat{\otimes} F_0 \end{array}$$

By the last result of the Introduction I_1 and I_2 are isomorphisms into; J' is an isomorphism into, since J is onto; similarly K_1 is an isomorphism. It must happen then that K_2 is an isomorphism into. Suppose we have $u = \sum_{i=1}^m f_K \otimes y_K$ in $E' \hat{\otimes} F$ and $\sum_{i=1}^m f_K(x)y_K$ in F_0 for each x in E . We shall show that there exists a u_0 in $J(E, F'_0)$ such that

$J'u_0 = L_1 I_1 u = u_1$. To show this we must show that $\langle \tilde{S}, u \rangle = 0$ where \tilde{S} is the Kernel of J , and the duality is that of $F' \hat{\otimes} E$ and $J(F', E)$. If $S \in F'_0 \hat{\otimes} E$, $||S|| \leq 1$, then there exists a constant C and an operator $\tilde{S} \in F' \hat{\otimes} E$, $||\tilde{S}|| \leq C$ such that $J\tilde{S} = S$. Let $\tilde{S} =$

$\sum_{i=1}^m \tilde{g}_{1i} \otimes x_{1i}$ be in $F' \hat{\otimes} E$ and such that $||J\tilde{S}|| \leq \epsilon$. If $\{x_{1i}\}_{i=1}^m \subseteq E$ and E an $\ell_{\infty, \lambda}$ space then there exists a subspace E_0 of E , $\{x_{1i}\} \subseteq E_0$, $d(E_0, \ell_{\infty}^n) < \lambda + \epsilon$, and there is an isomorphism $U: E_0 \rightarrow \ell_{\infty}^n$,

$||U|| \cdot ||U^{-1}|| < \lambda + \epsilon$, and we may assume $||U|| = 1$, $||U^{-1}|| < \lambda + \epsilon$.

Let $x_j = U^{-1}e_j$, $\{e_j\}_{j=1}^n$ the usual basis for ℓ_{∞}^n . We may write $x_{1i} = \sum_{j=1}^n t_{ij} x_j$, then

$$\sum_{i=1}^m \tilde{g}_{1i} \otimes x_{1i} = \sum_{i=1}^m \sum_{j=1}^n t_{ij} \tilde{g}_{1i} \otimes x_j = \sum_{j=1}^n \left(\sum_{i=1}^m t_{ij} \tilde{g}_{1i} \right) \otimes x_j.$$

Let $\tilde{g}_j = \sum_{i=1}^m t_{ij} \tilde{g}_{1i}$, and g_j equal the restriction of \tilde{g}_j to F_0 .

Then we have the following:

$$\begin{aligned} \epsilon &\geq ||J\tilde{S}|| = \sup_{\substack{y \in F_0 \\ ||y||=1}} ||Sy|| = \sup_{\substack{y \in F_0 \\ ||y||=1}} ||\sum_{j=1}^n g_j(y)x_j|| \\ &= \sup_{\substack{y \in F_0 \\ ||y||=1}} ||U^{-1} \left(\sum_{j=1}^n g_j(y)e_j \right)|| \end{aligned}$$

$$\begin{aligned}
& \geq \sup_{\substack{y \in F_0 \\ \|y\|=1}} \left\| \sum_{j=1}^n g_j(y) e_j \right\| = \max_j \sup_{\substack{y \in F_0 \\ \|y\|=1}} |g_j(y)| \\
& = \max_j \|g_j\|.
\end{aligned}$$

By the Hahn-Banach theorem, there exist \hat{g}_j in F' , $j=1, \dots, n$ $\hat{g}_j|_{F_0} = g_j$, $\|\hat{g}_j\| = \|g_j\| \leq \epsilon$. Let $\hat{S} = \sum_{j=1}^n \hat{g}_j \otimes x_j$; then we have that $\|\hat{S}\| = \|\hat{S} U U^{-1}\| \leq \|\hat{S} U\| \cdot \|U^{-1}\| < \epsilon(\lambda + \epsilon)$. Also, $\langle \hat{S}, u_1 \rangle = \langle \sum_{j=1}^n \hat{g}_j \otimes x_j, \sum_{K=1}^{\infty} f_K \otimes y_K \rangle = \sum_{j=1}^n \hat{g}_j \left(\sum_{K=1}^{\infty} f_K(x_j) y_K \right) = \sum_{j=1}^n g_j \left(\sum_{K=1}^{\infty} f_K(x_j) y_K \right) = \langle \tilde{S}, u_1 \rangle$. Thus we have that $|\langle \tilde{S}, u_1 \rangle| = |\langle \hat{S}, u_1 \rangle| \leq \epsilon(\lambda + \epsilon) \|u_1\|$; from which it follows that if $J\tilde{S} = 0$, then $\langle \tilde{S}, u_1 \rangle = 0$. Define u_2 , for $S \in F'_0 \hat{\otimes} E$, by $\langle S, u_2 \rangle = \langle J^{-1}S, u_1 \rangle$. Then u_2 is well defined and $\|u_2\| \leq C$. What we have proved at this point is this: if E is an L_∞ space and $T: E \rightarrow F$ is nuclear, then the restriction $T_0: E \rightarrow \overline{T(E)}$ is integral. If T is nuclear, we may represent it in the following form [7, p. 51]: $\sum \lambda_i f_i \otimes y_i$, $\{f_i\} \subseteq E'$, $\|f_i\| \rightarrow 0$, $\{y_i\} \subseteq F$, $\|y_i\| \rightarrow 0$, and $\sum_{i=1}^{\infty} |\lambda_i| < +\infty$. Consider the following diagram:

$$\begin{array}{ccc}
E & \xrightarrow{T} & F \\
R \downarrow & & \uparrow U \\
C_0 & \xrightarrow{S} & l_1
\end{array}$$

where

$$\begin{aligned}
R: E &\rightarrow C_0, & Rx &= \sum_{i=1}^{\infty} f_i(x) a_i \\
S: C_0 &\rightarrow l_1, & S\left(\sum_{i=1}^{\infty} t_i a_i\right) &= \sum_{i=1}^{\infty} \lambda_i t_i b_i \\
U: l_1 &\rightarrow F, & U\left(\sum_{i=1}^{\infty} s_i b_i\right) &= \sum_{i=1}^{\infty} s_i y_i
\end{aligned}$$

and $\{a_i\}$, $\{b_i\}$ denote, respectively, the usual bases for c_0 and ℓ_1 .

U is compact since the image of the unit ball of ℓ_1 is contained in the closed, convex, circled hull of the null sequence $\{y_i\} \subseteq F$.

S is obviously nuclear, so SR is nuclear. By the above, the astriction

$(SR)_0: E \rightarrow \overline{SR(E)}$ is integral, and if we denote by $U_0: \overline{SR(E)} \rightarrow \overline{USR(E)}$

the restriction and astriction of U , we have by [7, p. 132],

$U_0(SR)_0: E \rightarrow U(\overline{SR(E)})$ is nuclear, but $U(\overline{SR(E)}) = \overline{USR(E)} = \overline{T(E)}$ so T is fully nuclear.

A number of interesting corollaries follow from Theorem II.11.

Corollary II.12: The following are equivalent:

(1) E' has the approximation property and for each Banach space F , $N(E, F) = FN(E, F)$.

(2) E' has the approximation property and for each Banach space F , $N(E, F) = QN(E, F)$.

(3) E is isomorphic to a ℓ_∞ space.

Proof: (1) and (3) are equivalent from the theorem. If $N(E, \ell_\infty(S)) = FN(E, \ell_\infty(S))$ for each set S , then if $T: E \rightarrow F$ is quasi-nuclear, there exists [23, p. 52] a set S and an into isometry $I: F \rightarrow \ell_\infty(S)$ such that IT is nuclear [23], [25], hence it is fully nuclear, which is to say that T is nuclear. If $N(E, F) = QN(E, F)$ for all Banach spaces F , then if $T: E \rightarrow F$ is nuclear, its astriction $T_0: E \rightarrow \overline{T(E)}$ is certainly quasi-nuclear; and T_0 is nuclear.

The main examples of quasi-nuclear operators which are not nuclear are the non-nuclear Hilbert-Schmidt operators between Hilbert

spaces [24]. The existence of such operators for general Banach spaces appears not to have been studied. Thus the following corollary is interesting.

Corollary II.13: If E is not an \mathfrak{L}_∞ space, then there exists a Banach space F , and a quasi-nuclear operator $T:E \rightarrow F$ that is not nuclear.

Corollary II.14: Suppose E has an unconditional basis. Then E is isomorphic to c_0 if and only if for each Banach space F , $AS(E,F) = N(E,F)$.

Proof: If $AS(E,F) = N(E,F)$ then certainly $QN(E,F) = N(E,F)$, and by Corollary II.12 E is isomorphic to an \mathfrak{L}_∞ space and from [18 Corollary 5, p. 297] it follows that E must be isomorphic to c_0 . The converse is just II.9.

Corollary II.15: If E is isomorphic to a \mathfrak{L}_∞ space, and $T:E \rightarrow F$ is such that T' is nuclear, then T is nuclear.

Proof: If $T':F' \rightarrow E'$ is nuclear, then $T'':E'' \rightarrow F''$ is nuclear and if $J:E \rightarrow E''$ is the canonical injection, then $T'' J:E \rightarrow F''$ is nuclear. $T'' J$ must then be fully nuclear, but the image of $T'' J$ is isomorphic to the image of T , so T is nuclear.

Corollary II.16: If E is isomorphic to a quotient space of a \mathfrak{L}_∞ space G , and F is any Banach space, then there is an operator $T:E \rightarrow F$ that is not nuclear.

Proof: Construct $T:E \rightarrow F$ as in Theorem I. 20 . Let $S:G \rightarrow E$ be onto. If T is nuclear, then $TS:G \rightarrow F$ is nuclear, hence it is fully nuclear. Since $\overline{T(E)} = F_0$ is sufficiently Euclidean it follows that the dual $(TS)'_0:F'_0 \rightarrow G'$ of the astriction $(TS):G \rightarrow F_0$ is nuclear. But as in Theorem I. 20 $(TS)'_0$ is not even absolutely summing.

It is well known that if T is nuclear and the compositions ST and TR are defined, then both are nuclear [23, p. 47]. It is clear that ST is fully nuclear whenever T is fully nuclear. If we have, $R:E \rightarrow F$ and $T:F \rightarrow G$ such that $\overline{R(E)} = F$ then if T is fully nuclear then TR is fully nuclear. However, the next corollary and Remark II.2 (i) show that TR need not, in general, be fully nuclear whenever T is.

Corollary II. 17: Every nuclear operator is the restriction of a fully nuclear operator.

Proof: Let $T:E \rightarrow F$ be nuclear. Let S the unit ball of E' , and consider the canonical injection $J:E \rightarrow \ell_\infty(S)$, for suitable S . Then there exists a nuclear operator $\tilde{T}:\ell_\infty(S) \rightarrow F$ such that $T = \tilde{T}J$. \tilde{T} is fully nuclear, and T is the restriction of \tilde{T} .

Corollary II.18: Let $T:E \rightarrow F$ be nuclear such that $\dim T(E)$ is infinite. Then there exists an infinite dimensional subspace E_0 of E such that the restriction of T , $T_0:E_0 \rightarrow F$ is fully nuclear.

Proof: Suppose we represent T as $\sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i$, $\|f_i\| \rightarrow 0$, $\|y_i\| \rightarrow 0$, $\sum_{i=1}^{\infty} |\lambda_i| < +\infty$. Then consider the factorization of Theorem II.11:

$$\begin{array}{ccc}
 E & \xrightarrow{T} & F \\
 R \downarrow & & \uparrow U \\
 c_0 & \xrightarrow{S} & l_1
 \end{array}$$

Thus, $R(E)$ is a separable infinite dimensional subspace of c_0 , so $\overline{R(E)}$ contains a complemented subspace G isomorphic to c_0 [21],[31]. Let $P: \overline{R(E)} \rightarrow G$ be a projection. Let $E_0 = R^{-1}(G)$, E_0 is closed since G is closed, and E_0 is infinite dimensional since $R(E_0)$ is dense in G . Let $T_0: E_0 \rightarrow F$, $R_0: E_0 \rightarrow G$, $(US)_0: G \rightarrow F$ be the appropriate restrictions and restrictions. US is nuclear, so $(US)_0$ is nuclear, and G is isomorphic to c_0 , so $(US)_0: G \rightarrow F$ is fully nuclear. But $R_0(E_0)$ is dense in G so, $(US)_0 R_0: E_0 \rightarrow F$ is fully nuclear but $\overline{T(E_0)} = \overline{(US)_0 R_0(E_0)} = \overline{T_0(E_0)}$. Thus T_0 is fully nuclear.

We now prove the main result of this chapter. This is the analog of conjecture (C) of Grothendieck.

Theorem II.19: Let E and F be Banach spaces and suppose $\mathfrak{L}(E,F) = \text{FN}(E,F)$. Then E or F is finite dimensional. Of course, the converse is also true.

Proof: This follows easily from Theorem I. 20 . If E and F are infinite dimensional, by Theorem I. 13 there is a sufficiently Euclidean subspace F_0 of F and hence by Theorem I. 4 F_0' is sufficiently Euclidean. By Theorem I. 20 one may construct an operator

$T_0: E \rightarrow F_0$ such that $T'_0: F'_0 \rightarrow E'$ is not absolutely summing, hence not integral, so T_0 is not integral [7, § 7, sec. 3] and finally T_0 is not nuclear. If $J: F_0 \rightarrow F$ is the inclusion map, then $T = JT_0$ is not fully nuclear. Thus one of the spaces must be finite dimensional.

Remark II.20: If (C) is true as conjectured by Grothendieck, then any T in $\mathfrak{L}(E, F)$ must be fully nuclear. We feel that Theorem II.11 and the corollaries to Theorem II.11 can eventually be used to verify (C), stated in the following form: If there is a non fully-nuclear operator T from E to F , then there is a non-nuclear operator from E to F . We have not been able to overcome the obvious difficulties in proving this result.

We now state a result dual to Theorem II.19.

Theorem II. 21: If E and F are infinite dimensional Banach spaces, F isomorphic to a conjugate space, then there exists a quotient space G of F and an operator $T: E \rightarrow G$ such that T is not nuclear.

Proof: By hypothesis, there is a Banach space D such that D' is isomorphic to F . Let D_0 be a sufficiently Euclidean subspace of D . Then the adjoint of $J: D_0 \rightarrow D$, the canonical inclusion map, is onto: $J': D' \rightarrow D'_0$ hence we have an onto operator $S: F \rightarrow D'_0$ since F is isomorphic to D' . D_0 sufficiently Euclidean implies that D'_0 is the required space, and as above, there is a non nuclear operator $T: E \rightarrow G$.

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VITA

Charles Patrick Stegall, Jr. was born in Brookhaven, Mississippi on March 24, 1942. He attended public school in several states of the South, receiving no diplomas. He served in the United States Navy from March, 1960 to January, 1963, and was discharged from the United States Naval Reserve in 1967, with the rate of First Class Petty Officer. He received a B.S. from Louisiana State University, Baton Rouge, in May, 1965. At present he is a candidate for the degree of Doctor of Philosophy in the Department of Mathematics at Louisiana State University, Baton Rouge.

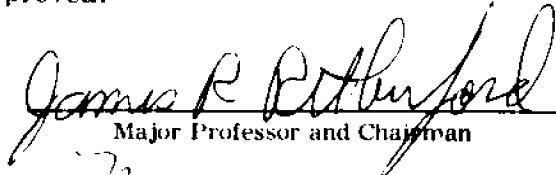
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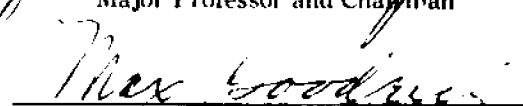
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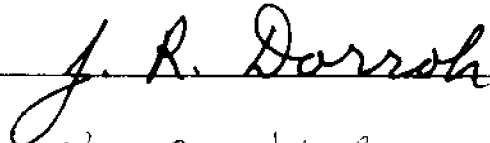
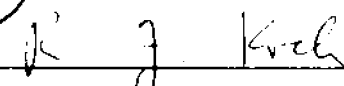

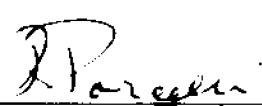
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